Chapter 9 interval estimation

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setup advantage of Interval estimation. coverage probability.

prob. that the interval covers the true param θ.

$$P_\theta(\theta \in (L(\mathbf{X}), U(\mathbf{X}))) \text{ or } P(\theta \in (L(\mathbf{X}), U(\mathbf{X})) \mid \theta)$$

the prob. statement refers to $\mathbf{X}$, not θ.

confidence interval = confidence coefficient + interval estimator

95% CI: (0.03, 0.53)

$\inf_\theta P_\theta(\theta \in (L(\mathbf{X}), U(\mathbf{X})))$ infimum.

methods of finding interval estimators

方法① Inverting a test stat hypothesis test & confidence interval are related

$A(\theta_0)$: accept region of a level α test of $H_0: \theta = \theta_0$.

$C(\mathbf{X})$ is a set in the parameter space.

fixed data/sample value find the param values. $C(\mathbf{X}) = \{\theta_0 : \mathbf{X} \in A(\theta_0)\}$ ☆

then $C(\mathbf{X})$ is a $1-\alpha$ confidence set.

hypothesis testing : fixed θ, find the acceptence/rejection region $A(\theta_0) = \{\mathbf{X} : \theta_0 \in C(\mathbf{X})\}$ is the acceptance region of a level α test of $H_0: \theta = \theta_0$.

example: $A(\theta_0) = \{x : \left(\frac{\sum x_i}{\theta_0}\right)^n e^{-\sum x_i/\theta_0} \geq k^*\}$

where k^* is chosen to satisfy:

$$P_\theta(X \in A(\theta_0)) \overset{?}{=} 1-\alpha \quad (\text{a level } \alpha \text{ test})$$

$$\Rightarrow C(\mathbf{X}) : \{\theta : \left(\frac{\sum x_i}{\theta_0}\right)^n e^{-\sum x_i/\theta_0} \geq k^*\}$$

方法② pivotal quantities

$X \sim f(\theta)$

a R.V. : $Q(X;\theta) = Q(x_1, x_2, \ldots, x_n, \theta)$ is a pivotal quantity if the PDF of $\theta(\mathbf{X}, \theta)$ is independent of all param, θ.

X	type	pivotal quantity
$f(x-\mu)$	location	$\bar{X}-\mu$
$\frac{1}{\sigma}f(\frac{x}{\sigma})$	scale	$\frac{\bar{X}}{\sigma}$
$\frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$	location-scale	$\frac{\bar{X}-\mu}{S}$

• $\text{Unif}(0,\beta) \to \text{CDF}: \left(\frac{x}{\beta}\right)^n$

pivot $\frac{X_{(n)}}{\beta} \sim \text{Beta}(n,1)$

• $X \sim \text{Beta}(\theta, 1) \to X^\theta \sim \text{Unif}(0,1)$

e.g. $X \sim N(\mu, \sigma^2)$ $t = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$,

t is a pivotal quantity, as t doesn't depend on μ, σ^2.

$$P_\theta(a \leq Q(\mathbf{X};\theta) \leq b) \geq 1-\alpha$$

for each θ_0, then

$$A(\theta_0) = \{\mathbf{X} : a \leq Q(\mathbf{X};\theta_0) \leq b\}$$

is the acceptance region for a level α test of $H_0: \theta = \theta_0$.

$\therefore$ $C(\mathbf{X}) = \{\theta_0 : a \leq Q(\mathbf{X};\theta_0) \leq b\}$ $1-\alpha$ CI.

example: $P(-a \leq \frac{\bar{X}-\mu}{S/\sqrt{n}} \leq a) = 1-\alpha$ ☆ ($\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$; acceptance region.)

$P(-a \leq \frac{\bar{X}-\mu}{S/\sqrt{n}} \leq a) = 1-\alpha$

$\therefore a = t_{n-1, \frac{\alpha}{2}}$

$\therefore -t_{n-1,\frac{\alpha}{2}} \leq \frac{\bar{X}-\mu}{S/\sqrt{n}} \leq t_{n-1,\frac{\alpha}{2}}$

$\Rightarrow \mu \in (\bar{X} \pm S/\sqrt{n} \cdot t_{(n-1),\frac{\alpha}{2}})$

$P(a \leq \frac{(n-1)S^2}{\sigma^2} \leq b) = 1-\alpha$ ($Q(X;\sigma^2) \sim \chi^2(n-1)$)

choose a, b that satisfy

$\chi^2(n-1)_{\frac{\alpha}{2}} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2(n-1)_{1-\frac{\alpha}{2}}$

$\therefore C(\mathbf{X}) : \frac{(n-1)S^2}{\chi^2(n-1)_{\frac{\alpha}{2}}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2(n-1)_{1-\frac{\alpha}{2}}}$ ■

① accept region $p = 1-\alpha$

② pivot include param

⇓

③ 通过 pivot 得到 pivot 的 bound

④ 得到 param 的 bound.

$$\underbrace{f(t \mid \theta)}_{\text{PDF of test stat } T} = g(Q(t,\theta)) \left| \frac{\partial Q(t,\theta)}{\partial t} \right|$$

if PDF of test stat T $f(t|\theta)$ can be written as some function g and monotone function Q, then $Q(t,\theta)$ is a pivot.